This short note is designed to supplement the "early transcendentals" approach to calculus by giving a definition of $exp(x)$ as a limit and computing its derivative. It is designed solely to provide complete details for the instructor. This proof is a bit easier than the usual proof because it uses powers of two instead of the positive integers in the definition of e^x as a limit.

Definition.

$$exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{2^n}\right)^{2^n}. \tag{1}$$

Proposition. *The limit in the definition above exists for all real numbers x and satisfies*

$$exp(x+y) = exp(x)exp(y)$$

and

$$\frac{d}{dx}exp(x) = exp(x).$$

Lemma 1. *For every real number x and every natural number n with $|x| < 2^{n-1}$, we have that*

$$\left(1+\frac{x}{2^{n-1}}\right)^{2^{n-1}} \le \left(1+\frac{x}{2^n}\right)^{2^n} \le \frac{1}{\left(1-\frac{x}{2^n}\right)^{2^n}} \le \frac{1}{\left(1-\frac{x}{2^{n-1}}\right)^{2^{n-1}}}. \tag{2}$$

Proof. For $|x| < 2^{n-1}$,

$$\left(1+\frac{x}{2^{n-1}}\right) \le \left(1+\frac{x}{2^n}\right)^2 \quad \text{and} \quad \left(1+\frac{x}{2^n}\right)\left(1-\frac{x}{2^n}\right) \le 1. \tag{3}$$

These inequalities are proved by multiplying out the two products. Raising the first inequality in (3) to the power 2^{n-1} proves the first inequality in (2), and replacing x by $-x$ proves the last inequality in (2). Raising the second inequality in (3) to the power 2^n proves the middle inequality in (2). $\square$

By Lemma 1, if $n \ge N$ and if $|x| < 2^N$ then $(1+\frac{x}{2^n})^{2^n}$ is increasing in n and bounded by $(1-\frac{x}{2^N})^{-2^N}$. Thus the limit in the definition exists. If $x = \varepsilon$, with $0 < |\varepsilon| < 1$, then setting $n = 1$ in the first and last inequalities of Lemma 1 and using induction we conclude

$$1+\varepsilon \le exp(\varepsilon) \le \frac{1}{1-\varepsilon}, \tag{4}$$

and hence for $\varepsilon > 0$,

$$1 \le \frac{exp(\varepsilon)-1}{\varepsilon} \le \frac{1}{1-\varepsilon},$$

and for $\varepsilon < 0$

$$1 \ge \frac{exp(\varepsilon)-1}{\varepsilon} \ge \frac{1}{1-\varepsilon}.$$

Thus

$$\lim_{\varepsilon\to 0}\frac{exp(\varepsilon)-1}{\varepsilon} = 1. \tag{5}$$

Lemma 2. *For real numbers x and y,*

$$exp(x+y) = exp(x)exp(y).$$

Proof. If $xy \geq 0$ and $2^n \geq |x| + |y|$, then

$$\left(1 + \frac{x}{2^n}\right)\left(1 + \frac{y}{2^n}\right) = \left(1 + \frac{x+y}{2^n} + \frac{xy}{2^{2n}}\right) \geq \left(1 + \frac{x+y}{2^n}\right).$$

Raising this inequality to the power 2^n and using (1) proves that $exp(x)exp(y) \geq exp(x+y)$. But also for $n \geq N$,

$$\left(1 + \frac{x+y}{2^n} + \frac{xy}{2^{2n}}\right) \leq \left(1 + \frac{x+y+\frac{xy}{2^N}}{2^n}\right).$$

Raising this inequality to the power 2^n and using (1) again proves that

$$exp(x)exp(y) \leq exp(x+y+\frac{xy}{2^N}) \leq exp(x)exp(y)exp(\frac{xy}{2^N}).$$

Letting $N \to \infty$ we obtain, by (4), $exp(x)exp(y) = exp(x+y)$ for all x, y satisfying $xy \geq 0$. If $xy < 0$, the inequalities just reverse, proving Lemma 2. □

Finally we conclude the proof of the Proposition using Lemma 2 and (5):

$$\lim_{\varepsilon \to 0} \frac{exp(x+\varepsilon) - exp(x)}{\varepsilon} = \lim_{\varepsilon \to 0} exp(x)\frac{exp(\varepsilon) - 1}{\varepsilon} = \exp(x)$$

□

Further properties:

i. Observe that $exp(x) > 0$ by Lemma 1, and $exp(0) = 1$. By the proposition $exp(x)$ is a continuous, increasing function.

ii. Let $\ln(x)$ denote the inverse function. Set $e = exp(1)$ and note $e > 2$, by (4). By Lemma 2, $exp(n) = e^n$ for integers n, and so $\lim_{x \to +\infty} exp(x) = \lim_{n \to +\infty} e^n = +\infty$. Similarly $\lim_{x \to -\infty} exp(x) = 0$. Thus $\ln(x)$ is defined for all positive numbers x. Since $exp(0) = 1$, we have that $\ln(1) = 0$.

iii. It follows from Lemma 2 that $\ln(xy) = \ln(x) + \ln(y)$.

iv. For all rational numbers r and all real numbers $a > 0$,

$$a^r = exp(r \ln(a)).$$

proof of iv. By Lemma 2 and induction, if $a > 0$ and if n is an integer, then $a^n = [exp(\ln a)]^n = exp(n \ln a)$. If m is also an integer, set $b = exp(\frac{n}{m} \ln a)$. Then $b^m = exp(m \cdot \frac{n}{m} \ln a) = a^n$, so that $a^{\frac{n}{m}} = b = exp(\frac{n}{m} \ln a)$. □

We extend this fact about rational numbers $\frac{n}{m}$ to all real numbers by definition.

Definition. *For real numbers x and a with $a > 0$, we define*

$$a^x = exp(x \ln(a)).$$

In particular, $e^x = exp(x)$.

By Lemma 2, $a^{x+y} = a^x a^y$.

Corollary.

$$\frac{d}{dx} a^x = a^x (\ln a) \quad \textit{and} \quad \frac{d}{dx} \ln(x) = \frac{1}{x}$$

Proof.

$$\frac{a^{x+h}-a^x}{h} = a^x\left(\frac{a^h-1}{h}\right) = a^x\left(\frac{e^{h\ln a}-1}{h\ln a}\right)\ln a.$$

Now let $h \to 0$ and apply (5). To find the derivative of $\ln x$, write

$$\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{exp(\ln(x+h))-exp(\ln(x))} = \frac{1}{exp(\ln(x))} = \frac{1}{x}.$$

The quantity in the second limit is the reciprocal of a difference quotient for the derivative of *exp* at the point $\ln x$. □

D. Marshall
January 2012